

POLYNOMIALS THAT ARISE IN A POLYA URN

GAMBLING GAME

by

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ABSTRACT

A gambling game based on the Polya urn process is discussed. Working with the expected value of the game produces an interesting sum. The main result of the paper is this sum is a polynomial with degree equal to the initial number of balls in the urn.

CHAPTER I

INTRODUCTION

1.1 Overview

This paper focuses on computing the expected value for a gambling game based on the Polya urn process. It is an extension of one of the games presented in an earlier paper by Martin and Ho [10]. It is useful to recall how the Polya urn process works. At the start of play, the Polya urn contains b black balls and w white balls. A ball is drawn and it is replaced by two balls of the same color. This process is then repeated.

We develop a sum S which plays an important role in computing the value of the game. The main result of the paper is that this sum is a polynomial with degree $b + w$, the initial number of balls in the urn.

The next section provides background information on the Polya urn process and the motivating paper [10]. Chapter 2 defines the game under consideration in this paper. Chapter 3 analyzes the value of the game and develops the aforementioned sum S . Chapter 4 consists of examples of computing S in particular cases. The main result, S is a polynomial of degree $b + w$, is contained in Chapter 5. Chapter 6 concludes.

1.2 Background

1.2.1 Polya Urn Process

The Polya urn process is described in the paper by Eggenberger and Polya [11]: an urn contains balls of two different colors and a player draws balls from the urn. After each draw, the selected ball is returned together with s balls of the same color (in our paper, $s = 1$). The Polya urn process is used to model a number of phenomena including: economic activity, learning, and the spread of diseases.

As discussed in [10], the Polya urn process can be used to model a two-commodity

market without advertising. The typical example is the competition in the personal computer market between PCs and Macs. Assume that the two products have (nearly) equal quality. A new user's decision to buy will depend on the current relative share of the market; if there are more PC users, the new user will be more likely to buy a PC. This is reasonable because the company with the larger market share will have more software developed for it and more support options. Several papers by Arthur also discuss modeling of economic activity with urn models ([1],[2],[3],[4]).

The replacement learning model is described in [9]. This model uses a generalized Polya urn process to describe a subject engaged in a learning experiment. On each trial he attempts to perform a task, after each attempt he gains more experience. The urn represents the subject's knowledge or ability. Black balls correspond to correct actions and white balls to incorrect actions. In the model, each trial has two stages: performance and learning. In the first stage, a ball is selected from the urn (and returned). If black is selected, the subject correctly performs the task, otherwise he does not. In the second stage, k balls are randomly selected from the urn and replaced with k black balls. Learning is regarded as the process of replacing white balls with black balls. The following references discuss other learning models involving the Polya urn process: [5], [7], [9], and [12].

Applications to the spread of infectious disease are described in [8] and [9].

1.2.2 Motivating Paper

As mentioned above, this paper is an extension of an earlier paper by Martin and Ho [10]. The original paper discusses two gambling games based on the Polya urn scheme. This paper considers the second game.

The first game involves computing the value of information in an urn game. There are two urns in the game: the primary urn starts with b black balls and w white balls, the secondary urn starts empty. The dealer draws a ball from the primary urn and replaces it with two balls of the same color. He also places a ball of the same color

in the secondary urn. The dealer draws in this fashion k times. Hence, at the end of this random setup, the primary urn contains $b + w + k$ balls and the secondary urn contains k balls. The player guesses the color of the next ball to be drawn and receives \$1 for a correct guess. However, before guessing, the player can purchase information: he can pay to see $s \leq k$ of the balls in the secondary urn (these s balls will be drawn randomly from the urn). The paper calculates the expected value of this information. Interestingly, it is discovered that the information may have negative, zero, or positive value. It is also shown that in many cases the additional information will not affect the player's betting strategy.

In the second game a player pays to play N times in succession. On each draw he guesses the color of the ball and is paid \$1 for a correct guess. The objective is to compute the expected value of this game. Martin and Ho compute the expected value in the case where the urn starts with one black and one white ball. They note that the computation for general initial conditions is difficult and “essentially the same as recovering the input sequence for a nonlinear stochastic process.”

CHAPTER II
PROBLEM FORMULATION

This paper deals with game 2 from [10]. In this game a player guesses the color of the balls drawn from a Polya urn. The urn starts with b black balls and w white balls; balls are drawn and replaced using the Polya urn scheme. At the start of the game, the player pays a fee to play a fixed number of rounds N . The player knows the initial distribution (b, w) and he is told the color of the ball after each draw (before making his next guess). He is given \$1 for each correct guess.

Let X_j and Y_j be the number of black and white balls, respectively, at the end of draw j . Also, let $V(N)$ be the expected value of the game with N draws. We have the following result for $V(N)$,

Theorem 1. *In a game with $X_0 = b$ and $Y_0 = w$, the value is given by*

$$V(N) = \sum_{j=0}^{N-1} \frac{E(\max\{X_j, Y_j\})}{j + b + w}$$

Proof. The player knows the initial distribution (b, w) . So, on the first draw he knows the values of X_0 and Y_0 and clearly he'll bet black iff $b \geq w$. Since there are a total of $b + w$ balls in the urn, his probability of winning on the first draw is $\frac{\max(X_0, Y_0)}{b+w}$ (this is also his expected payoff because he gets \$1 for a correct guess). After the first draw, he is told the color of the ball picked, and thus he knows the values of X_1 and Y_1 . Using the same argument, his expected payoff on the second draw is $\frac{\max(X_1, Y_1)}{b+w+1}$. Continuing this argument gives

$$V(N) = E\left(\sum_{j=0}^{N-1} \frac{(\max\{X_j, Y_j\})}{j + b + w}\right)$$

The result follows from the linearity of expectations. □

CHAPTER III
DERIVING THE SUM S

In this chapter we simplify the above expression for $V(N)$. The sum S arises in the process.

Lemma 1. *In the case where $X_0 = b$ and $Y_0 = w$,*

$$E(\max\{X_n, Y_n\}) = \sum_{j=0}^n Pr(n, n-j|b, w) \max\{n-j+b, j+w\}$$

where

$$Pr(n, k|b, w) = \binom{n}{k} \frac{(b+k-1)!(w+n-k-1)!(b+w-1)!}{(b+w+n-1)!(b-1)!(w-1)!}$$

Proof. The possible values of (X_n, Y_n) are $(n-j+b, j+w)$ for $0 \leq j \leq n$. Let $Pr(n, k|b, w)$ be the probability of getting k black balls in n draws, given $X_0 = b$ and $Y_0 = w$. Then by definition,

$$E(\max\{X_n, Y_n\}) = \sum_{j=0}^n Pr(n, n-j|b, w) \max\{n-j+b, j+w\}$$

Now, let us compute $Pr(n, k|b, w)$. Every sequence of k black balls and $n-k$ white balls has the same probability (see [9], [10]). Consider the sequence BB...BWW...W, k black draws followed by $n-k$ white draws. Clearly, the probability of the sequence is given by

$$\left[\frac{b}{b+w} \frac{b+1}{b+w+1} \cdots \frac{b+k-1}{b+w+k-1} \right] \left[\frac{w}{b+w+k} \frac{w+1}{b+w+k+1} \cdots \frac{w+n-k-1}{b+w+n-1} \right] = \frac{(b+k-1)!(w+n-k-1)!(b+w-1)!}{(b+w+n-1)!(b-1)!(w-1)!}$$

There are $\binom{n}{k}$ sequences of k black balls and $n-k$ white balls, and thus the result is proven. □

Theorem 1 combined with Lemma 1 shows that $V(N)$ can be computed analytically for any initial distribution (b, w) . However, we are still interested in simplifying the expression for $V(N)$.

Define the following:

$$\begin{aligned} f(j) &\equiv f(j; n, b, w) \equiv \frac{(j+w-1)!}{j!} \cdot \frac{(n-j+b-1)!}{(n-j)!} \\ C &\equiv C(n, b, w) \equiv \frac{(b+w-1)!}{(b-1)!(w-1)!} \cdot \frac{n!}{(n+b+w-1)!} \end{aligned}$$

We have $Pr(n, n-j|b, w) = Cf(j)$. Without loss of generality, let $b \geq w$. Thus,

$$\begin{aligned} E(\max\{X_n, Y_n\}) &= \sum_{j=0}^n Pr(n, n-j|b, w) \max\{n-j+b, j+w\} \\ &= \sum_{j=0}^n w Pr(n, n-j|b, w) + C \sum_{j=0}^n f(j) \max\{n-j+b-w, j\} \\ &= w + C \sum_{j=0}^n f(j) \max\{n-j+(b-w), j\} \end{aligned}$$

For the last equality, note that $\sum_{j=0}^n Pr(n, n-j|b, w) = 1$, since the indices cover all possible numbers of black balls on n draws.

Therefore, a key component in simplifying $V(N)$ is simplifying the following sum:

$$S \equiv \sum_{j=0}^n f(j) \max\{n-j+(b-w), j\}$$

It is useful to define $c = b - w$. The following theorem is a generalization of Lemma 6.1 in [10].

Theorem 2. Let $S = \sum_{j=0}^n f(j) \max\{n-j+c, j\}$.

Case I. $n < c$

$$S = \sum_{j=0}^n f(j)(n-j+c)$$

Case II. $n \geq c$ then if $n + c$ is odd,

$$S = \sum_{j=0}^{c-1} f(j)(n - j + c) + \sum_{j=\frac{n+c+1}{2}}^n \left[f(j) + f(n - j + c) \right] j$$

and if $n + c$ is even,

$$S = \sum_{j=0}^{c-1} f(j)(n - j + c) + \left(\frac{n+c}{2} \right) \cdot f\left(\frac{n+c}{2} \right) + \sum_{j=\frac{n+c+2}{2}}^n \left[f(j) + f(n - j + c) \right] j$$

Remark. For notational convenience, in Case I, we refer to S as S_1 . In Case II, we refer to S as S_o or S_e depending on whether $n + c$ is odd or even, respectively. Case I is the degenerate case, consequently we focus mainly on Case II.

Proof. Consider Case I, where $n < c$. Clearly, $n + c > 2n \geq 2j$ for $0 \leq j \leq n$. Therefore, over the same j indices, $\max\{n - j + c, j\} = n - j + c$. This gives the result.

Now consider Case II, i.e., let $n \geq c$. Then $n + c \geq 2c > 2j$ for $0 \leq j \leq c - 1$, thus $\max\{n - j + c, j\} = n - j + c$ over the same indices. Hence:

$$S = \sum_{j=0}^{c-1} f(j)(n - j + c) + \sum_{j=c}^n f(j) \max\{n - j + c, j\}$$

Define $(a, b) \equiv \max(a, b)$.

Let $n + c$ be odd and consider the sequence $(n - j + c, j)$, $c \leq j \leq n$. There is an index j such that $n - j + c = j + 1$ (i.e. the value of two adjacent terms is equal). Solving this equality gives, $j = \frac{n+c-1}{2}$. Clearly the sequence starts at (n, c) and because j increases by 1 we have:

$$\begin{aligned} & (n, c); (n - 1, c + 1); \dots; \left(\frac{n + c + 1}{2}, \frac{n + c - 1}{2} \right); \\ & \left(\frac{n + c - 1}{2}, \frac{n + c + 1}{2} \right); \dots; (c + 1, n - 1); (c, n) \end{aligned}$$

Notice that for the first half of the sequence, $c \leq j \leq \frac{n+c-1}{2}$, $(n-j+c, j) = n-j+c$, and then for the second half of the sequence, $(n-j+c, j) = j$. Also note that the values in the sequence repeat (backwards) after the term¹ $j = \frac{n+c-1}{2}$. Moreover, for $\frac{n+c+1}{2} \leq j \leq n$, term j has the same value as term $n-j+c$ (namely j). Hence:

$$\sum_{j=c}^n f(j) \max\{n-j+c, j\} = \sum_{j=\frac{n+c+1}{2}}^n \left[f(j) + f(n-j+c) \right] j$$

The case where $n+c$ is even is almost identical. Except here, there is an index j such that $n-j+c = j$. Solving this equality gives $j = \frac{n+2}{2}$. Hence the sequence $(n-j+c, j)$, $c \leq j \leq n$ is given by:

$$(n, c); (n-1, c+1); \dots; \left(\frac{n+c}{2} + 1, \frac{n+c}{2} - 1\right); \left(\frac{n+c}{2}, \frac{n+c}{2}\right) \\ \left(\left(\frac{n+c}{2} + 1, \frac{n+c}{2} - 1\right); \dots; (c+1, n-1); (c, n)\right)$$

Using a similar argument as in the odd case gives,

$$\sum_{j=c}^n f(j) \max\{n-j+c, j\} = \left(\frac{n+c}{2}\right) \cdot f\left(\frac{n+c}{2}\right) + \sum_{j=\frac{n+c}{2}+1}^n \left[f(j) + f(n-j+c) \right] j$$

□

¹We do not refer to the term (n, c) as the first term in the sequence but the c^{th} term. We label the terms by the value of j .

CHAPTER IV

EXAMPLES: COMPUTING S USING THEOREM 2

This chapter computes examples using Theorem 2 (in the case where $n \geq c$). Many of the computations in this chapter were done in Maple.

4.1 Case when $(b, w) = (1, 1)$

In addition to computing S in the case of $(1, 1)$, this section also provides a correction to a slight mistake in the original paper [10]. Here $c = 0$ and $f(j) = 1$, thus $S = \sum_{j=0}^n \max\{n - j, j\}$. Using Theorem 2 in the n even case gives:

$$\begin{aligned} S_e &= \frac{n}{2} + 2 \sum_{j=\frac{n}{2}+1}^n j \\ &= \frac{n}{2} + 2 \left(\sum_{j=0}^n j - \sum_{j=0}^{\frac{n}{2}} j \right) \\ &= \frac{3}{4}n^2 + n \end{aligned}$$

Doing similar calculations for n odd gives, $S_o = \frac{3}{4}n^2 + n + \frac{1}{4}$

This specific case of Theorem 2 is Lemma 6.1 from [10]. There is a slight mistake in the statement of this lemma. The sum was calculated correctly, but expanded incorrectly.

$$\begin{aligned} S_e &= \frac{n(3n+4)}{4} \neq \frac{3}{4}(n+1)^2 - \frac{3}{2}(n+1) + \frac{7}{4} \\ S_o &= \frac{(3n+1)(n+1)}{4} \neq \frac{3}{4}(n+2)^2 - \frac{1}{2}(n+1) \end{aligned}$$

This incorrect expansion was used in computing $V(n+1)$ (Theorem 6.3), and hence there is also an error in that result. The correct formula—found using the above, Lemma 1, and Theorem 1—is given by

$$V(n+1) = \sum_{j=0}^n \frac{1}{j+2} + \sum_{j \text{ even} \leq n} \frac{j(3j+4)}{4(j+1)(j+2)} + \sum_{j \text{ odd} \leq n} \frac{(3j+1)}{4(j+2)}$$

4.2 Case when $(b, w)=(2,1)$

Here $c = 1$ and $f(j) = n - j + 1$. By Theorem 2,

$$\begin{aligned} S_o &= \sum_{j=0}^0 (n - j + 1)^2 + \sum_{j=\frac{n+2}{2}}^n (n + 1)j \\ &= (n + 1)^2 + 3/8 n^3 + 5/8 n^2 + 1/4 n \\ &= \frac{3n^3 + 13n^2 + 18n + 8}{8} \end{aligned}$$

$$\begin{aligned} S_e &= (n + 1)^2 + \left(\frac{n+1}{2}\right)^2 + \sum_{j=\frac{n+1}{2}+1}^n (n + 1)j \\ &= (n + 1)^2 + \left(\frac{n+1}{2}\right)^2 + \frac{3}{8}(n^3 + n^2 - n - 1) \\ &= \frac{3n^3 + 13n^2 + 17n + 7}{8} \end{aligned}$$

4.3 Case when $(b, w)=(5,2)$

Here $f(j) = (j + 1)(n - j + 4)(n - j + 3)(n - j + 2)(n - j + 1)$. This expands to

$$j^5 - 4j^4n + 6j^3n^2 - 4j^2n^3 + jn^4 - 9j^4 + 26j^3n - 24j^2n^2 + 6jn^3 + n^4 + 25j^3 - 40j^2n + 5jn^2 + 10n^3 - 15j^2 - 20jn + 35n^2 - 26j + 50n + 24$$

Consider the two sums needed for computing S_o

$$\begin{aligned} \sum_{j=0}^2 f(j)(n + 3 - j) &= 6n^5 + 38n^4 + 114n^3 + 202n^2 + 192n + 72 \\ \sum_{j=\frac{n+4}{2}}^n \left[f(j) + f(n + 3 - j) \right] j &= \frac{47}{1920} n^7 + \frac{391}{640} n^6 + \frac{313}{960} n^5 - \frac{93}{32} n^4 - \frac{479}{240} n^3 \\ &\quad + \frac{37}{20} n^2 + \frac{6}{5} n \end{aligned}$$

Combining the above, we get S_o is given by

$$\frac{47}{1920} n^7 + \frac{391}{640} n^6 + \frac{6073}{960} n^5 + \frac{1123}{32} n^4 + \frac{26881}{240} n^3 + \frac{4077}{20} n^2 + \frac{966}{5} n + 72$$

Similar work can be done to find S_e .

4.4 Case when $b = w$

When $b = w$, it is easy to compute the sums using Maple.¹ The values of S_o for $b = w = k$, $1 \leq k \leq 6$, are included in the appendix. In all of the above cases, S is a polynomial in n of degree $b + w$.

The next chapter proves that this result is true in general.

¹When $b > w$, Maple can still be used to compute the sums, but it is not as straightforward.

CHAPTER V
MAIN THEOREM

We now state the main theorem of the paper,

Theorem 3. *S in Theorem 2 is a polynomial in n of degree b + w*

Section 4.1 shows that the theorem is true when $b = w = 1$. Throughout this chapter, we assume that $b > 1$.¹

This theorem is proved below in several parts. The first section establishes several preliminary results. In the second section, we prove that the auxiliary terms which appear in Case II, $\sum_{j=0}^{c-1} f(j)(n - j + c)$ and $\binom{n+c}{2} \cdot f\left(\frac{n+c}{2}\right)$, are polynomials in n with degree less than $b + w$. Finally, in the third section we prove that the main sum for S_o is a polynomial in n with degree $b + w$ (the same basic argument works for the main sums of S_1 and S_e). The main sums are:

$$\begin{aligned} S_1 & : \quad \sum_{j=0}^n f(j)(n - j + c) \\ S_o & : \quad \sum_{j=\frac{n+c+1}{2}}^n \left[f(j) + f(n - j + c) \right] j \\ S_e & : \quad \sum_{j=\frac{n+c+2}{2}}^n \left[f(j) + f(n - j + c) \right] j \end{aligned}$$

5.1 Preliminary Results

Definition. *Let $\mathbf{x} \in \mathbb{R}^n$ and let $\phi(\mathbf{x})$ be a function which is a linear combination of products of the coordinates of \mathbf{x} . Then ϕ is called a **multivariable polynomial**.*

Example. The following are multivariable polynomials

- a. $\phi(x) = 3x^3 + 4x^2 + 3$
- b. $\psi(x, y) = 3x^3 + 4x^2y + 5xy^2 + y^3 + 2x + 7y$

¹Since $b \geq w$ all the cases are handled.

c. $\eta(x, y, z) = xyz + x^2y^2 + x^5z^2$

We define an operator L which picks out the highest degree terms of multivariable polynomials.

Definition. Let ϕ be a multivariable polynomial with degree p . The operator L is defined by

$L(\phi) =$ sum of the terms in ϕ with degree p .

Example.

a. Consider $\phi(x) = 3x^3 + 4x^2 + 3$. Then $L[\phi(x)] = 3x^3$.

b. Consider $\psi(x, y) = 3x^3 + 4x^2y + 5xy^2 + y^3 + 2x + 7y$. Then

$L[\psi(x, y)] = 3x^3 + 4x^2y + 5xy^2 + y^3$

The following three facts are obvious, and thus are stated without proof,

Lemma 2. Let $m \in \mathbb{N}$ be fixed and let $x \in \mathbb{N}$ be indeterminate. Consider $p(x) = \frac{(x+m)!}{x!}$. Then $p(x)$ is a polynomial in x with degree m and leading coefficient $a_m = 1$.

Lemma 3. Let $\phi_1(\mathbf{x})$ and $\phi_2(\mathbf{y})$ be multivariable polynomials. If $\phi = \phi_1\phi_2$ then ϕ is a multivariable polynomial in (\mathbf{x}, \mathbf{y}) and

$$\begin{aligned} L[\phi(\mathbf{x}, \mathbf{y})] &= L[\phi_1(\mathbf{x})\phi_2(\mathbf{y})] \\ &= L[\phi_1(\mathbf{x})] \cdot L[\phi_2(\mathbf{y})] \end{aligned}$$

Lemma 4. Let $\phi_1(\mathbf{x}), \dots, \phi_n(\mathbf{x})$ be multivariable polynomials in \mathbf{x} all with degree p . Define $\phi(\mathbf{x}) = \sum_{i=1}^n \phi_i(\mathbf{x})$. Then

1. $\phi(\mathbf{x})$ has degree $\leq p$

2. If $\sum_{i=1}^n L[\phi_i] \neq 0$, $\phi(\mathbf{x})$ has degree p and

$$L[\phi] = L\left[\sum_{i=1}^n \phi_i\right] = \sum_{i=1}^n L[\phi_i]$$

Example.

a. Consider $\phi_1(x) = 3x^3 + 4x^2 + 3$, $\phi_2(x) = -3x^3 + 10x^2 + 5$. Then

$$L[\phi_1 + \phi_2] = 14x^2, \text{ but } L[\phi_1] + L[\phi_2] = 0.$$

b. Consider $\psi_1(x) = x^5 + 7$, $\psi_2(x) = 8x^5 + 12x^2 + 11$. Then

$$L[\psi_1 + \psi_2] = L[\psi_1] + L[\psi_2] = 9x^5.$$

Remark. Below we somewhat loosely use $L[\sum_{i=1}^n \phi_i] = \sum_{i=1}^n L[\phi_i]$. But after the fact, we do check that $\sum_{i=1}^n L[\phi_i] \neq 0$

Lemma 5. $f(j) \equiv f(j, n)$ is a multivariable polynomial in (j, n) with²

$$L[f(j, n)] = \sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i n^{b-1-i} j^{w-1+i}$$

Proof. Let $\phi_1(j) = \frac{(j+w-1)!}{j!}$ and $\phi_2(n-j) = \frac{(n-j+b-1)!}{(n-j)!}$. By Lemma 2, ϕ_1 is a polynomial in j with $L[\phi_1(j)] = j^{w-1}$ and ϕ_2 is a polynomial in $(n-j)$ with $L[\phi_2(n-j)] = (n-j)^{b-1}$. By definition, $f(j, n) = \phi_1(j)\phi_2(n-j)$ and thus, by Lemma 3, $f(j, n)$ is a multivariable polynomial with

$$\begin{aligned} L[f(j, n)] &= j^{w-1}(n-j)^{b-1} \\ &= \sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i n^{b-1-i} j^{w-1+i} \end{aligned}$$

The last equality comes from expanding $(n-j)^{b-1}$ using the binomial theorem. \square

5.2 Auxiliary Terms

Proposition 1. For a fixed value of c , $\sum_{j=0}^{c-1} f(j)(n-j+c)$ is a polynomial in n with degree $\leq \max\{b, b+w-2\}$.

Proof. It is helpful to make the following definitions,

$$\begin{aligned} \phi_1(n, j) &\equiv f(j)(n-j+c) = nf(j) - (j-c)f(j) \\ \phi_2(n) &\equiv \sum_{j=0}^{c-1} f(j)(n-j+c) = \sum_{j=0}^{c-1} \phi_1(n, j) \end{aligned}$$

²Based on the notation in this section we should refer to $f(j)$ as $f(j, n)$

From Lemmas 3, 4, and 5,

$$L[\phi_1(n, j)] = \sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i \left[n^{b-i} j^{w-1+i} - n^{b-1-i} j^{w+i} \right] \quad (5.1)$$

Now consider $\phi_2(n) = \sum_{j=0}^{c-1} \phi_1(n, j)$. For a fixed j , $0 \leq j \leq c-1$, $\phi_1(n, j) = \phi_1(n)$ is a polynomial in n . From (5.1) it appears that when j is fixed, $\phi_1(n)$ has degree b (let $i = 0$). However, we need to be careful. When viewing ϕ_1 as a polynomial in (n, j) , it has degree $b+w-1$. Hence, there might be a term αn^{b+w-2} which would not be included in the result of $L[\phi_1(n, j)]$. Thus, for a fixed j , when ϕ_1 is a polynomial in n , the term αn^{b+w-2} needs to be considered. We have, for a fixed j ,

$$\text{degree}(\phi_1(n)) \leq \max\{b, b+w-2\}$$

The result follows using Lemma 4 since $\phi_2(n) = \sum_{j=0}^{c-1} \phi_1(n, j)$ □

Proposition 2. $f\left(\frac{n+c}{2}\right) \cdot f\left(\frac{n+c}{2}\right)$ is a polynomial in n with degree $\leq b+w-1$

Proof. Using Lemma 5,

$$L\left[f\left(\frac{n+c}{2}\right)\right] = \sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i n^{b-1-i} \left(\frac{n+c}{2}\right)^{w-1+i}$$

Thus, $f\left(\frac{n+c}{2}\right)$ is a polynomial in n with degree $b+w-2$. Multiplying by $\left(\frac{n+c}{2}\right)$ yields the result. □

5.3 Main Sum

The work in this section is all done for S_o . The same procedures can be used to handle S_1 and S_e (these results are proved in the appendix).

The following result is well known (see [6]).

Fact 1. Let $k \geq 0$. $\sum_{i=1}^n i^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + O(n^{k-1})$

Corollary 1.

$$\sum_{j=\frac{n+c+1}{2}}^n j^k = g(k)n^{k+1} + O(n^k),$$

where

$$g(k) = \frac{2^{k+1} - 1}{(k+1)2^{k+1}}$$

Proof.

$$\sum_{j=\frac{n+c+1}{2}}^n j^k = \sum_{j=1}^n j^k - \sum_{j=1}^{\frac{n+c-1}{2}} j^k$$

Using Fact 1 on both of the above sums and simplifying gives the result. \square

Proposition 3.

$$L[S_o] = \left(\sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i g(w+i) + \sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i g(b+i) \right) n^{b+w}$$

Proof. From Lemmas 3 and 5,

$$L[jf(j)] = \sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i n^{b-1-i} j^{w+i}$$

For a fixed j , $\frac{n+c+1}{2} \leq j \leq n$, $jf(j)$ is a polynomial in n with degree $b+w-1$.

Thus, by Lemma 4,

$$\begin{aligned} L\left[\sum_{j=\frac{n+c+1}{2}}^n jf(j)\right] &= \sum_{j=\frac{n+c+1}{2}}^n L[jf(j)] \\ &= \sum_{j=\frac{n+c+1}{2}}^n \left[\sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i n^{b-1-i} j^{w+i} \right] \\ &= \sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i n^{b-1-i} \left[\sum_{j=\frac{n+c+1}{2}}^n j^{w+i} \right] \end{aligned}$$

$$= \left[\sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i g(w+i) \right] n^{b+w}$$

The last equality uses Corollary 1.

A similar argument shows that

$$L \left[\sum_{j=\frac{n+c+1}{2}}^n j f(n-j+c) \right] = \left[\sum_{i=0}^{w-1} \binom{w-1}{i} (-1)^i g(b+i) \right] n^{b+w}$$

Recall, from Theorem 2,

$$S_o = \sum_{j=0}^{c-1} f(j)(n-j+c) + \sum_{j=\frac{n+c+1}{2}}^n \left[f(j) + f(n-j+c) \right] j$$

Combining Proposition 1 and Lemma 4 with the above gives the result. □

It remains to show that coefficient in Proposition 3 is nonzero. It is sufficient to show the following

Proposition 4. *Let $H(m, a) = \sum_{i=0}^m \binom{m}{i} (-1)^i g(a+i)$. Then*

$$H(m, a) > 0 \quad \forall a, m \in \mathbb{N}$$

Proof.

$$\begin{aligned} H(m, a) &= \sum_{i=0}^m \binom{m}{i} (-1)^i \left(\frac{1}{a+i+1} - \frac{1}{2^{a+i+1}} \cdot \frac{1}{a+i+1} \right) \\ &= \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{1}{a+i+1} - \sum_{i=0}^m \binom{m}{i} (-1)^i \left(\frac{1}{2^{a+i+1}} \cdot \frac{1}{a+i+1} \right) \end{aligned}$$

Let $\phi_1(x) \equiv \sum_{i=0}^m \binom{m}{i} (-1)^i x^{a+i}$. Then

$$\int_0^1 \phi_1(x) dx = \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{1}{a+i+1}$$

Also,

$$\begin{aligned}\phi_1(x) &= x^a \sum_{i=0}^m \binom{m}{i} (-1)^i x^i \\ &= x^a (1-x)^m \quad [\text{Binomial Theorem}]\end{aligned}$$

Let $\phi_2(x) \equiv \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{x^{a+i}}{2^{a+i+1}}$. Then

$$\int_0^1 \phi_2(x) dx = \sum_{i=0}^m \binom{m}{i} (-1)^i \left(\frac{1}{2^{a+i+1}} \cdot \frac{1}{a+i+1} \right)$$

Also,

$$\begin{aligned}\phi_2(x) &= \frac{x^a}{2^{a+1}} \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{x^i}{2^i} \\ &= \frac{x^a}{2^{a+1}} \left(1 - \frac{x}{2}\right)^m \quad [\text{Binomial Theorem}]\end{aligned}$$

Combining the above yields,

$$\begin{aligned}H(m, a) &= \int_0^1 \phi_1(x) dx - \int_0^1 \phi_2(x) dx \\ &= \int_0^1 x^a (1-x)^m dx - \frac{1}{2^{a+1}} \int_0^1 x^a \left(1 - \frac{x}{2}\right)^m dx \\ &= \int_0^1 x^a (1-x)^m dx - \int_0^{\frac{1}{2}} x^a (1-x)^m dx \\ &= \int_{\frac{1}{2}}^1 x^a (1-x)^m dx\end{aligned}$$

The second to last equality follows by substituting $r = \frac{x}{2}$ in the second integral. Clearly, $x^a(1-x)^m > 0$ for $\frac{1}{2} < x < 1$ and thus

$$H(m, a) = \int_{\frac{1}{2}}^1 x^a (1-x)^m dx > 0$$

□

Propositions 3 and 4 prove Theorem 3 in the case of S_o . Using the results in Sections 5.1 and 5.2 and applying the same basic techniques as in Section 5.3 proves the theorem in the other two cases.

CHAPTER VI
CONCLUDING REMARKS

The investigation of the expected value of a Polya urn gambling game leads to a complicated sum involving factorials and maximums. First we reduced the sum to one that does not involve maximums (Theorem 2), then we proved that this sum is a polynomial of degree $b + w$, the initial number of balls in the urn (Theorem 3).

BIBLIOGRAPHY

- [1] Arthur, W.B. Self-Reinforcing Mechanisms in Economics, in *The Economy as an Evolving Complex System*, 1988, Addison-Wesley, Redwood City, Ca.
- [2] Arthur, W.B. Competing Technologies and Lock-in by Historical Small Events, *Economic Journal*, 99(1989), 116-131.
- [3] Arthur, W.B. Industry Location Patterns and the Importance of History, *Mathematical Social Sciences*, 19(1990), 235-251.
- [4] Arthur, W.B. *Increasing Returns and Path Dependence in the Economy*, 1994, Univ of Michigan Press, Ann Arbor.
- [5] Audley, R.J., Jonckheere, A.R. A stochastic description of the learning behaviour of an individual subject, *Q.J. Exp. Psychol.*, 9(1956), 87-94,
- [6] Beardon, A.F. Sums of Powers of Integers, *American Mathematical Monthly*, 103 (1996), 201-213.
- [7] Bush, R.R., Mosteller, F. *Stochastic Models for Learning*, 1955, Wiley, New York.
- [8] Feller, W. *An Introduction to Probability Theory and its Applications, Vo1.1*, 1950, Wiley, London.
- [9] Johnson, N.L., Kotz, S, *Urn models and their application: an approach to modern discrete probability theory*, 1977, Wiley, New York.
- [10] Martin, C.F., Ho, Y.C. Value of Information in the Polya Urn Process, *Inform. Sci.*, 147(2002), no. 1-4, 65-90.
- [11] Eggenberger, F., Polya, G. Uber die Statistik verketteter Vorgange, *Z. Angew. Math. Mech.*, 1(1923), 279-289.
- [12] Restle,F., Greeno, J.G., *Introduction to Mathematical Psychology*, 1970, Addison Wesley, Reading, Ma.
- [13] Petkovsek, Wilf, and Zeilberger, *A=B*, 1996, A K Peters, Wellesley, Ma.

APPENDIX A

MAPLE CODE

The sums for $c = 0$ can easily be computed using the Maple sum command `gospers`. The code below is used to compute S_c in the case where $b = w$ (where b would be specified).

```
> f:=(j,b,w)->(j+w-1)!*(n-j+b-1)!/(j!*(n-j)!);
> with(sumtools):
> simplify(gospers((f(j,b,b)+f(n-j,b,b))*j, j=(n+1)/2..n));
```

The code produces the following results for $b = w = k$, $1 \leq k \leq 6$.

1. $\frac{3}{4}n^2 + n + \frac{1}{4}$
2. $\frac{11}{96}n^4 + \frac{3}{4}n^3 + \frac{77}{48}n^2 + \frac{5}{4}n + \frac{9}{32}$
3. $\frac{1}{320}(3+n)(7n^5 + 89n^4 + 398n^3 + 746n^2 + 555n + 125)$
4. $\frac{1}{35840}(3+n)(163n^7 + 4215n^6 + 43551n^5 + 228867n^4 + 643161n^3 + 928533n^2 + 603365n + 128625)$
5. $\frac{1}{322560}(3+n)(n+7)(319n^8 + 11480n^7 + 169576n^6 + 1326560n^5 + 5909206n^4 + 14967800n^3 + 20272224n^2 + 12724560n + 2679075)$
6. $\frac{1}{5677056}(3+n)(n+7)(1255n^{10} + 71666n^9 + 1760871n^8 + 24347304n^7 + 207948174n^6 + 1132919004n^5 + 3922760294n^4 + 8325358216n^3 + 10006964331n^2 + 5845355010n + 1188616275)$

APPENDIX B

MAIN SUMS

This appendix contains proofs that the main sums for S_e and S_1 are both polynomials in n with degree $b + w$.

Main sum for S_e

Corollary 2.

$$\sum_{j=\frac{n+c+2}{2}}^n j^k = g(k)n^{k+1} + O(n^k),$$

where

$$g(k) = \frac{2^{k+1} - 1}{(k+1)2^{k+1}}$$

Proof.

$$\sum_{j=\frac{n+c+2}{2}}^n j^k = \sum_{j=1}^n j^k - \sum_{j=1}^{\frac{n+c}{2}} j^k$$

Using Fact 1 on both of the above sums and simplifying gives the result. □

Corollaries 1 and 2 imply

$$L\left[\sum_{j=\frac{n+c+1}{2}}^n j^k\right] = L\left[\sum_{j=\frac{n+c+2}{2}}^n j^k\right]$$

Using this fact and following the proof of Proposition 3, yields

$$L[S_e] = L[S_o]$$

Main sum for S_1

Proposition 5.

$$L[S_1] = \left(\sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i \frac{1}{(w+i)(w+i+1)} \right) n^{b+w}$$

The proof of this fact is almost identical to the proof involving the main sum of S_o . Although, here we can use Fact 1 directly.

Proof. From Lemmas 3 and 5,

$$L[nf(j)] = \sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i n^{b-i} j^{w-1+i}$$

For a fixed j , $0 \leq j \leq n$, $nf(j)$ is a polynomial in n with degree $b + w - 1$. Thus, by Lemma 4,

$$\begin{aligned} L\left[\sum_{j=0}^n nf(j)\right] &= \sum_{j=0}^n L[nf(j)] \\ &= \sum_{j=0}^n \left[\sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i n^{b-i} j^{w-1+i} \right] \\ &= \sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i n^{b-i} \left[\sum_{j=0}^n j^{w-1+i} \right] \\ &= \left[\sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i \frac{1}{w+i} \right] n^{b+w} \end{aligned}$$

The last equality uses Fact 1.

A similar argument shows that

$$L\left[\sum_{j=0}^n jf(j)\right] = \left[\sum_{i=0}^{b-1} \binom{b-1}{i} (-1)^i \frac{1}{w+1+i} \right] n^{b+w}$$

Lemma 4 with the above gives the result. □

It remains to show that the coefficient in Proposition 5 is nonzero. It is sufficient to show

Proposition 6. *Let $G(m, a) = \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{1}{(a+i)(a+i+1)}$. Then*

$$G(m, a) > 0 \quad \forall a, m \in \mathbb{N}$$

Proof.

$$G(m, a) = \sum_{i=0}^m \binom{m}{i} (-1)^i \left(\frac{1}{a+i} - \frac{1}{a+i+1} \right)$$

Let $\phi_1(x) \equiv \sum_{i=0}^m \binom{m}{i} (-1)^i x^{a-1+i}$. Then

$$\int_0^1 \phi_1(x) dx = \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{1}{a+i}$$

Also,

$$\begin{aligned} \phi_1(x) &= x^a \sum_{i=0}^m \binom{m}{i} (-1)^i x^i \\ &= x^{a-1} (1-x)^m \quad [\text{Binomial Theorem}] \end{aligned}$$

Similarly, let $\phi_2(x) \equiv \sum_{i=0}^m \binom{m}{i} (-1)^i x^{a+i}$, and we get

$$\begin{aligned} \int_0^1 \phi_2(x) dx &= \sum_{i=0}^m \binom{m}{i} (-1)^i \frac{1}{a+i+1} \\ \phi_2(x) &= x^a (1-x)^m \end{aligned}$$

Combining the above yields,

$$\begin{aligned} G(m, a) &= \int_0^1 \phi_1(x) dx - \int_0^1 \phi_2(x) dx \\ &= \int_0^1 x^{a-1} (1-x)^m dx - \int_0^1 x^a (1-x)^m dx \\ &= \int_0^1 (x^{a-1} - x^a) (1-x)^m dx \\ &= \int_0^1 x^{a-1} (1-x)^{m+1} dx \end{aligned}$$

Clearly, $x^{a-1}(1-x)^{m+1} > 0$ for $0 < x < 1$ and thus the result follows. □

This proof also illustrates that the leading coefficient of S_1 is given by

$$\int_0^1 x^{w-1} (1-x)^b dx$$

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