POWER SERIES COEFFICIENTS OF SOME CLASSICAL FUNCTIONS

by

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ABSTRACT

In this thesis we consider several problems pertaining to extremal conditions arising in Geometric Function Theory. First, we extend the known extremal conditions of a particular function space to a slightly more general space and determine the condition of equality. Next, we discuss the current status of the Krzyż conjecture with a new observation. In the following chapter, we develop a mechanism to translate an extremal function of one space into another space and apply it to a special case of the Krzyż conjecture. The last chapter of the paper is devoted to discussing the current status of Brannan's conjecture with some observations that might lead to its proof.

CHAPTER 1

INTRODUCTION

Much effort of Complex Analysts consists of determining results about analytic functions which are classically devoid of pathologies typically found in Real Analysis or Topology. The context of this paper is in Geometric Function Theory where among other things, determining inequalities involving the power series coefficients of analytic functions whose ranges satisfy some geometric condition is of major interest. Several powerful theorems led up to the development of Geometric Function Theory. One of the most important theorems that marked its beginning was the Riemann Mapping Theorem which states: Let D be a simply connected domain which is a proper subset of the complex plane and $z_0 \in D$. Then, there is a unique function f which maps D conformally onto the unit disk and has the properties $f(z_0) = 0$ and $f'(z_0) > 0$. In the plane we think of simply connected regions as regions without any holes; therefore, one typically restricts his/her study to univalent functions which preserve the simple connectedness.

Due to this result, one who is interested in a particular geometric configuration in the plane might benefit from analyzing its associated function space. One hopes that such a function space is normal, in which case its closure is compact. This is useful because continuous extremals, such as the modulus of power series coefficients, are guaranteed to attain their extremal values on compact spaces. If an extremal is attained by some function in the space of interest, then additional insight might be gained about the space by analyzing the extremal function. Some geometric configurations lead to functions spaces that have nice analytic representations from which further information can be obtained. In addition, most of the geometric configurations that are studied are highly symmetric leading to rather elegant extremal conditions, which was the initial motivation of this study for the author.

CHAPTER 2

SOME FUNCTION SPACES AND INITIAL GAP FUNCTIONS

Let $\mathcal{S}$ denote the class of functions f, which are analytic and univalent in the unit disk $\mathbb{D}$ and are normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, and $\mathcal{P}$ denote the class of analytic functions p which have positive real part in $\mathbb{D}$ and are normalized by $p(0) = 1$. Let Σ denote the class of functions analytic and univalent in the the exterior of $\overline{\mathbb{D}}$, except for a simple pole at infinity with residue 1, and $\mathcal{B}_0$ denote the class of analytic functions f in the unit disk $\mathbb{D}$ satisfying $0 < |f(z)| \leq 1$.
A set E is said to be starlike with respect to a point $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. A starlike function is a conformal mapping of the unit disk onto a region that is starlike with respect to the origin.

Let $\mathcal{S}^*$ denote the class of functions in $\mathcal{S}$ which are starlike, Σ^* denote the class of functions in Σ that map the complement of $\overline{\mathbb{D}}$ to a set whose complement is starlike with respect to the origin, $\mathcal{S}_0^*$ be the class of analytic functions of the form $f(z)/z$ for $f \in \mathcal{S}^*$, and Σ_0^* be the subclass of Σ^* whose functions have zero as their constant coefficient. The interested reader should read Duren's text [8] for detailed explanations of these function spaces excluding possibly $\mathcal{B}_0$, which is less discussed in the literature.

For a given space of analytic functions H, let $M(H, n, z_0) = \{h \in H \mid f \in H$ implies $|b_n| \leq |a_n|$ for $h(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ and $f(z) = \sum_{k=0}^{\infty} b_k(z - z_0)^k\}$, which is the set of functions in H whose n^{th} coefficient is extremal in modulus and let $H^{-1} = \{\widetilde{h} \mid \widetilde{h} = 1/h \text{ for some } h \in H\}$.

We are mainly interested in the extremal conditions of some of the function spaces introduced during the next few sections. For $\mathcal{S}$, the Koebe function $K(z) = z(1-z)^{-2}$ serves as an extremal function, which satisfies the following:

Theorem 2.0.1. *Let* $f \in \mathcal{S}$. *For each* λ *such that* $0 \neq |\lambda| = r < 1$, f *satisfies the* *following inequalities. Equality occurs if and only if f is a suitable rotation of $K(z)$.*

(1) The range of every function of $\mathcal{S}$ contains the disk $\{\alpha \mid |\alpha| < \frac{1}{4}\}$ (Koebe One-Quarter Theorem).

(2) $\dfrac{1-r}{(1+r)^3} \le |f'(\lambda)| \le \dfrac{1+r}{(1-r)^3}$ *(Distortion Theorem).*

(3) $\dfrac{r}{(1+r)^2} \le |f(\lambda)| \le \dfrac{r}{(1-r)^2}$ *(Growth Theorem).*

(4) $\dfrac{1-r}{1+r} \le \left|\dfrac{\lambda f'(\lambda)}{f(\lambda)}\right| \le \dfrac{1+r}{1-r}$.

(5) For $f(z) = z + \displaystyle\sum_{n=1}^{\infty} a_n z^n \in S$, $|a_n| \le n$ *for* $n = 1, 2, \ldots$ *(de Branges' theorem).*

Miller's text [11] demonstrates how (ii)-(iv) can be reformulated in terms of subordination.

2.1. Functions of Positive Real Part

Besides the unit disk, the right half-plane is the most natural region one encounters from elementary conformal mappings. Due to the Herglotz Representation for functions of positive real part and the ease at which variational methods can be applied to them, the properties of $\mathcal{P}$ can be useful in several situations. Indeed, we will use the extremality conditions for $\mathcal{P}$ given below to prove a result about a different function space in the next section.

Theorem 2.1.1 (Carathéodory's Lemma)**.** *Let $p \in \mathcal{P}$ such that $p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k$. Then $|a_k| \le 2$ for $k = 1, 2, \ldots$. Equality was completely characterized by M. S. Robertson [14] and can be given as follows.*

Suppose $a_n = 2$ and $|a_k| < 2$ for $k = 1 \ldots n-1$. Then,

$$p(z) = \frac{1 + \sum_{k=1}^{n-1} p_k z^k + z^n}{1 - z^n} = 1 + \sum_{k=1}^{n-1} p_k \sum_{j=0}^{\infty} z^{k+nj} + 2 \sum_{k=1}^{\infty} z^{nk}$$

such that $p_m = \overline{p}_{n-m}$ for $1 \le m \le n-m$ and $p_m = 2\sum_{k=1}^{n} \lambda_k e^{2mk\pi i/n}$ for $0 \le \lambda_k \le 1$, $\sum_{k=1}^{n} \lambda_k = 1$, $m = 1, \ldots, n$.

2.2. Starlike Functions

Starlike functions have enough structure to investigate phenomena that are less tractable in $\mathcal{S}$. An explicit example of this was the verification of a subcase of de Branges' Theorem (for starlike functions) which can be found in Duren's text [8]. Considering that subcase was possible since the Koebe function is starlike. In addition, the structure of $\mathcal{S}^*$ brings about properties that are interesting in their own right. Among these is the rather amazing bijective relationship with $\mathcal{P}$.

Theorem 2.2.1. *Let f be analytic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then, $f \in S^*$ if and only if $zf'(z)/f(z) \in \mathcal{P}$.*

A theorem of importance to our initial investigation was created by Clunie [7].

Theorem 2.2.2. *The n^{th} coefficient of every function in Σ_0^* satisfies $|b_n| \leq 2/(n+1)$ with equality only for rotations of $K(z^{-n-1})^{-1/(n+1)}$ such that $K(z)$ is the Koebe function.*

The next theorem's proof, which answers a conjecture by K. Wirths is largely based on Clunie's proof for the previous theorem with the main insights of the author consisting of translating from Σ_0^* to $[\mathcal{S}_0^*]^{-1}$ and proving the case of equality.

Theorem 2.2.3. *Let $f \in S^*$, $g(z) = z/f(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \in [\mathcal{S}_0^*]^{-1}$. Then, for every $n \in \mathbb{N}$, $|a_n| \leq 2/n$ with equality if and only if g is of the form $(1+\omega z^n)^{2/n}$ or $1 + 2\lambda\omega z + \omega^2 z^2$ such that $\lambda \in [-1, 1]$ and $|\omega| = 1$.*

Proof. Let $h(z) = zg(1/z) = 1/f(1/z) = z + \sum_{k=0}^{\infty} b_k z^{-k}$ for $|z| > 1$.

$$\frac{\omega f'(\omega)}{f(\omega)} = \omega h(1/\omega)\left(\frac{h'(1/\omega)}{\omega^2 h^2(1/\omega)}\right) = \frac{zh'(z)}{h(z)}$$

maps $\{z \mid |z| > 1\}$ to the right half plane and ∞ to one.

Hence,

$$\frac{\dfrac{zh'(z)}{h(z)} - 1}{\dfrac{zh'(z)}{h(z)} + 1} = \frac{zh'(z) - h(z)}{zh'(z) + h(z)}$$

maps $\{z \mid |z| > 1\}$ into the unit disk sending ∞ to zero, which allows us to appeal to the Maximum Modulus theorem to determine that $|\Psi| < 1$ for $\Psi(z) := z\dfrac{zh'(z) - h(z)}{zh'(z) + h(z)}$ in $|z| > 1$ since it is analytic at ∞. Upon expanding the numerator and denominator of Ψ we have,

$$z^2h'(z) - zh(z) = z^2\left[1 - \sum_{k=0}^{\infty} kb_k z^{-k-1}\right] - z\left[z + \sum_{k=0}^{\infty} b_k z^{-k}\right] = -\sum_{k=0}^{\infty}(k+1)b_k z^{-k+1},$$

and

$$\begin{aligned} zh'(z) + h(z) &= z\left[1 - \sum_{k=0}^{\infty} kb_k z^{-k-1}\right] + \left[z + \sum_{k=0}^{\infty} b_k z^{-k}\right] \\ &= 2z - \sum_{k=0}^{\infty}\left[kb_k - b_k\right] z^{-k}. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\left(2 - \sum_{k=0}^{n-1}\left[kb_k + b_k\right] z^{-k-1}\right)\Psi(z) \\ &= \left(-\sum_{k=0}^{\infty}(k+1)b_k z^{-k+1}\right)\left(\frac{1}{z(1 + \sum_{k=n+1}^{\infty} c_k z^{-k})}\right) \\ &\left(-\sum_{k=0}^{\infty}(k+1)b_k z^{-k}\right)\left(1 + \sum_{k=n+1}^{\infty} d_k\right) = -\sum_{k=0}^{n}(k+1)b_k z^{-k} + \sum_{k=n+1}^{\infty} e_k z^{-k} \end{aligned}$$

for some sequences $\{c_k\}$, $\{d_k\}$, and $\{e_k\}$.

Now, we wish to employ a common use of Parseval's formula: if $|f(z)| \leq |g(z)|$ for analytic $f(z) = \sum_{k=0}^{\infty} f_k z^k$ and $g(z) = \sum_{k=0}^{\infty} g_k z^k$ in the unit disk, then $\sum_{k=0}^{\infty} |f_k|^2 \leq \sum_{k=0}^{\infty} |g_k|^2$. It is clear this works because

$$\sum_{k=0}^{\infty} |f_k|^2 = \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leq \lim_{r\to 1^-} \frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} |g_k|^2.$$

This application also works if the functions are analytic on the complement of $\overline{\mathbb{D}}$ in which case r would approach one from the right.

Since $|\Psi(z)| < 1$,

$$4+\sum_{k=0}^{n-1}|kb_k-b_k|^2 \geq \sum_{k=0}^{n}|(k+1)b_k|^2+\sum_{k=n+1}^{\infty}|e_k|^2 \geq \sum_{k=0}^{n}|(k+1)b_k|^2.$$

Therefore,

$$\begin{aligned}|(n+1)b_n|^2 &\leq 4+\sum_{k=0}^{n-1}\left[(k-1)^2-(k+1)^2\right]|b_k|^2\\ &= 4-4\sum_{k=0}^{n-1}k|b_k|^2 = 4-4\sum_{k=1}^{n-1}k|b_k|^2 \leq 4,\end{aligned}$$

with equality for $n \geq 2$ only if $b_1 = \cdots = b_{n-1} = 0$ which implies that $|b_n| \leq 2/(n+1)$. Since g and h have the same power series coefficients with the index moved up by one due to the z factor, $|a_n| \leq 2/n$ with equality for $n \geq 3$ only if $a_2 = \cdots = a_{n-1} = 0$. For the case of equality, let $g(z) = 1 + a_1 z + \sum_{k=n}^{\infty} a_k z^k$ such that $|a_n| = 2/n$. Then,

$$\frac{zf'(z)}{f(z)} = 1 - \frac{zg'(z)}{g(z)} \in \mathcal{P}.$$

Hence,

$$\Phi(z) := \cfrac{1}{1-\cfrac{zg'(z)}{g(z)}} \in P.$$

Upon expanding Φ we have

$$\begin{aligned}\Phi(z) &= \frac{g(z)}{g(z)-zg'(z)} = \frac{1+a_1z+\sum_{k=n}^{\infty}a_kz^k}{(1+a_1z+\sum_{k=n}^{\infty}a_kz^k)-z(a_1+\sum_{k=n}^{\infty}ka_kz^{k-1})}\\ &= \frac{1+a_1z+\sum_{k=n}^{\infty}a_kz^k}{1-\sum_{k=n}^{\infty}(k-1)a_kz^k}.\end{aligned}$$

Hence,

$$\Phi(z) = (1+a_1z+a_nz^n+O(z^{n+1}))(1+(n-1)a_nz^n+O(z^{n+1})) \text{ as } z \to 0$$

because

$$\frac{1}{1-\sum_{k=n}^{\infty}c_kz^k} = 1+c_nz^n+O(z^{n+1}) \text{ as } z \to 0.$$

Thus,

$$\begin{aligned}\Phi(z) &= (1 + a_1 z + a_n z^n + O(z^{n+1}))(1 + (n-1)a_n z^n + O(z^{n+1})) \\ &= 1 + a_1 z + n a_n z^n + O(z^{n+1}) \text{ as } z \to 0.\end{aligned}$$

Since $|na_n| = 2$ and $\Phi \in P$, we can appeal to Robertson's [14] characterization of extremal functions in $\mathcal{P}$,

$$\Phi(z) = \frac{1 + \sum_{k=1}^{n-1} p_k z^k + z^n}{1 - z^n} = 1 + \sum_{k=1}^{n-1} p_k z^k + 2z^n + O(z^{n+1})$$

upon precomposing Φ with an appropriate rotation. The p_k satisfy $p_{n-k} = \overline{p}_k$ for $0 < k < n$. For $n > 2$, $a_1 = p_1 = \overline{p}_{n-1} = 0$, which implies that

$$\Phi(z) = \frac{1 + \omega z^n}{1 - \omega z^n}.$$

For $n = 2$,

$$\Phi(z) = \frac{1 + a_1 \omega z + \omega^2 z^2}{1 - \omega^2 z^2},$$

and $a_1 \in [-2, 2]$ also by Robertson [14].

$$\begin{aligned}1/(1 - z(\log(1 + \omega z^n)^{2/n})') &= \frac{1 + \omega z^n}{1 - \omega z^n} \text{ and} \\ 1/(1 - z(\log(1 + 2\lambda\omega z + \omega^2 z^2))') &= \frac{1 + 2\lambda\omega z + \omega^2 z^2}{1 - \omega^2 z^2}.\end{aligned}$$

Since our proposed forms for the case of equality satisfies the necessary normalization and differential equations, the case of equality has been determined. □

2.3. Bounded Nonvanishing Functions

The following conjecture has eluded several mathematicians due its failure to yield to classical techniques.

Let $F(z) = \exp\left((z-1)/(z+1)\right)$.

Conjecture 2.3.1 (Krzyż Conjecture)**.** *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}_0$. *Then,* $|a_k| \leq 2/\mathrm{e}$ *for* $k \in \mathbb{N}$ *with equality if and only if* $f(z) = \lambda F(\omega z^k)$ *for some* $|\lambda| = |\omega| = 1$.

Known results for the Krzyż Conjecture:

- Hummel, et al. [9] observed that there exists an extremal function for each coefficient since $\mathcal{B}_0$ is a normal family, which is compact when unioned with $f(z) \equiv 0$.

- Samaris [16] has recently extended Szapiel approach to show Krzyż Conjecture is known for $n \leq 5$.

- Let $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathcal{B}_0$. Since $|f(z)| < 1$, the same application of Parseval's Theorem as in Theorem 2.2.3 implies $\sum_{k=0}^{\infty} |a_k|^2 \leq 1$.

- Hummel, et al. [9] observed that if $f \in M(\mathcal{B}_0, n, 0)$, then $f(z) = \mathrm{e}^{i\alpha} \exp\left(- \sum_{k=1}^{m} \lambda_k \frac{1 + B_k z}{1 - B_k z} \right)$ where $|B_k| = 1$, $\lambda_k > 0$, and $1 \leq m \leq n$.

- According to Szapiel [18], $4/5 + (4/\pi)\sin(\pi/20) \approx .9991785471$ is a known uniform bound for all $n > 1$.

- Hummel, et al. showed that $F(z^n)$ provides a strict local maximum for Re a_n among suitably normalized functions in $\mathcal{B}_0$.

- Lewandowski and Szynal considered $f(z) = \exp(-tp(z))$ for some $p(z) = \sum_{k=0}^{\infty} p_k \in \mathcal{P}$ to established the formula $a_n = [(-1)^n/n!]e^{-t} D_n$, where D_n is the determinant of an $n \times n$ matrix with elements ktp_k for $1 \leq k \leq n$. They used this to show that $|a_n| \leq 2/e$ if $p_2, \ldots, p_{n-1} = 0$.

Lemma 2.3.2. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k \in M(B_0, n, 0)$. *Then,* $|a_0|, \ldots,$ *and* $|a_{[n/2]}| \leq \min(\frac{|a_n|}{2}, \frac{\sqrt{5}}{5})$.

Proof. Consider

$$\exp\left(t \frac{\omega z^m - 1}{\omega z^m + 1} \right) f(z) = [e^{-t} + 2e^{-t} t \omega z^m + O(z^{2m})] \sum_{k=0}^{\infty} a_k z^k =$$

$$\mathrm{e}^{-t} \sum_{k=0}^{m-1} a_k z^k + \mathrm{e}^{-t} \sum_{k=m}^{2m-1} [a_k + 2t\omega a_{k-m}] z^k + O(z^{2m}) \text{ as } z \to 0.$$

Since multiplication of functions in $\mathcal{B}_0$ is closed, and $n = m, \dots, 2m-1$ if and only if $m = n - \left[\frac{n}{2}, \dots, n\right]$ if and only if $n - m = 0, \dots, \left[\frac{n}{2}\right]$, we have that $\mathrm{e}^{-t}|a_n + 2t\omega a_j| \leq |a_n|$ for each $j = 0, \dots, \left[\frac{n}{2}\right]$, all positive t, and $|\omega| = 1$. Let $j = 0, \dots, \left[\frac{n}{2}\right]$ and $\omega = \mathrm{e}^{i(\arg a_n - \arg a_j)}$. For every $t > 0$, $\mathrm{e}^{-t}|a_n + 2t\omega a_j| = \mathrm{e}^{-t}\left(|a_n| + 2t|a_j|\right) \leq |a_n|$, i.e., $0 \leq e^t - 1 - \left(2\frac{|a_j|}{|a_n|}\right)t = \left(1 - 2\frac{|a_j|}{|a_n|}\right)t + O(t^2)$ as $t \to 0^+$. Hence, it is necessary that $2|a_j| \leq |a_n|$. Now, $|a_j|^2 + (2|a_j|)^2 \leq |a_j|^2 + |a_n|^2 \leq \sum_{k=0}^{\infty} |a_k|^2 \leq 1$; hence, $|a_j| \leq \frac{\sqrt{5}}{5}$ for each $j = 0, \dots, \left[\frac{n}{2}\right]$. $\square$

At this point one would like to show that $|a_n| = 2|a_0|$ is necessary for extremality, which in turn would give the new bound $|a_n| \leq \frac{2\sqrt{5}}{5}$ and probably simplify some variational techniques. However, it does not seem likely that the method used here can be extended to give better results since the only general way for $|a_n| < |b_k a_0 + b_0 a_n|$ is for $|b_0|$ to be near one which in turn drives $|b_k|$ to zero. The functions $\left(\frac{1}{2}(1+z)\right)^n = \frac{1}{2^n} + \frac{n}{2^n}z + O(z^2)$ as $z \to 0$ are members of $\mathcal{B}_0$ that show the necessity of f being extremal in Lemma 2.3.2.

2.4. Initial Gap Functions

We shall say an analytic function $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ is an initial n^{th}-gap function about z_0 if $a_1 = \cdots = a_{n-1} = 0$.

Theorem 2.4.1. *Let H be a space of analytic functions such that for every $h \in H$, the base point z_0 is in the domain of h and $h(z_0) = z_1$ for some fixed z_1. Let p be an analytic function such that for each $h \in H$, $p \circ h$ is well-defined, and p is univalent in some neighborhood U of z_1. Let $H^* = \{h^* | h^* = p \circ h \text{ for some } h \in H\}$. If $f \in M(H, z_o, n)$ and $g \in M(H^*, z_0, n)$ are initial n^{th}-gap functions, then $p \circ f \in M(H^*, z_0, n)$ and $p^{-1} \circ g \in M(H, z_0, n)$.*

Proof. We appeal to the formula of Faà di Bruno [17]: If $r(t)$ and $s(t)$ are functions for which all the necessary derivatives are defined then

$$D^n r(s(t)) = \sum \frac{n!}{k_1! \cdots k_n!} (D^k r)(s(t)) \left(\frac{Ds(t)}{1!}\right)^{k_1} \cdots \left(\frac{D^n s(t)}{n!}\right)^{k_n}$$

where $k = k_1 + \cdots + k_n$ and the sum is over all $k_1, \ldots, k_n$ for which $k_1 + 2k_2 + \cdots + nk_n = n$. If s is an initial n^{th}-gap function about z_0 and $k_j \neq 0$ for any $j < n$, then $D^m r(s(z_0)) = 0$.

Hence, upon expanding $p \circ f$ as a series we have

$$p \circ f(z) = p(z_1) + p'(z_1) \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n + O(z^{n+1}) \text{ as } z \to z_0.$$

By assumption $g = p \circ \widetilde{f}$ for some $\widetilde{f} \in H$. $\widetilde{f}^{-1}(U)$ is some neighborhood of z_0 and $p^{-1} \circ g|_{\widetilde{f}^{-1}(U)} = (p^{-1} \circ p) \circ \widetilde{f}|_{\widetilde{f}^{-1}(U)} = \widetilde{f}|_{\widetilde{f}^{-1}(U)}$. Now similarly,

$$p^{-1} \circ g(z) = z_1 + (p^{-1})'(p(z_1)) \frac{g^{(n)}(z_0)}{n!} (z - z_0)^n + O(z^{n+1}) \text{ as } z_0 \to 0.$$

Consider $\widetilde{H} = \{\widetilde{h}(z) | \widetilde{h}(z) = h|_{V(h)}(z)$ such that $V(h)$ is a region contained in the domains of $h(z)$ and $f^{-1}(U)$ for $h(z) \in H\}$. The expansion of $\widetilde{h}(z)$ about z_0 is the same as the expansion of its associated $h(z)$. Since $f|_{V(h)} \in M(\widetilde{H}, z_0, n)$, $p^{-1} \circ g(z) \in \widetilde{H}$, $g(z) \in M(H^*, z_0, n)$, and $p \circ f \in H^*$ we have the respective conditions

$$\left| (p^{-1})'(p(z_1)) \frac{g^{(n)}(z_0)}{n!} \right| \leq \left| \frac{f^{(n)}(z_0)}{n!} \right| \text{ and } \left| p'(z_1) \frac{f^n(z_0)}{n!} \right| \leq \left| \frac{g^n(z_0)}{n!} \right|.$$

Local univalence gives $p'(z_1)((p^{-1})'(p(z_1))) = 1$ which implies

$$\left| \frac{g^{(n)}(z_0)}{n!} \right| \leq \left| p'(z_1) \frac{f^{(n)}(z_0)}{n!} \right| \leq \left| \frac{g^{(n)}(z_0)}{n!} \right| \text{ and}$$
$$\left| \frac{f^{(n)}(z_0)}{n!} \right| \leq \left| (p^{-1})'(p(z_1)) \frac{g^{(n)}(z_0)}{n!} \right| \leq \left| \frac{f^{(n)}(z_0)}{n!} \right|.$$

□

CHAPTER 3

POSSIBLE USES FOR THE RESULT ABOUT INITIAL GAP FUNCTIONS

A natural use of Theorem 2.4.1 in the previous section arises when the members of H are nonvanishing in a simply connected region, in which case we can use $f(z) = \log z$ or $g(z) = z^\alpha$ as our locally univalent functions. This theorem provides an easy way to verify the Krzyż Conjecture for the n^{th} expansion term of an initial n^{th}-gap function.

Proof. Indeed, let $t_0 > 0$ and $B_0^n(t_0)$ be the initial n^{th}-gap functions of the form $\exp(-t_0 p(z))$ for $p \in \mathcal{P}$ and $\mathcal{P}(t_0) = \{p \mid p = -\log \circ f \text{ for } f \in B_0^n(t_0)\}$. There exists a $f \in M(B_0^n(t_0), n, 0)$ since a subset of a normal space is normal and a sequence of initial n^{th}-gap functions must converge to an initial n^{th}-gap function. We can write

$$t_0 \frac{1+z^n}{1-z^n} = t_0 + 2t_0 \Sigma_{k=1}^{\infty} z^{nk} \in M(\mathcal{P}(t_0), n, 0)$$

by applying Carathéodory's Lemma to functions of the form $p(z)/t_0$ for $p \in \mathcal{P}(t_0)$, i.e., functions in $\mathcal{P}$. Hence, we can use Theorem 2.4.1 to see that

$$\exp\left(-t_0 \frac{1+z^n}{1-z^n}\right) = \mathrm{e}^{-t_0} - 2t_0 \mathrm{e}^{-t_0} z^n + O(z^{n+1}) \in M(B_0^n(t_0), n, 0) \text{ as } z \to 0.$$

Consequently,

$$2t\mathrm{e}^{-t} \leq \frac{2}{\mathrm{e}} \text{ for all } t > 0 \text{ since } \frac{d}{dt} t\mathrm{e}^{-t} = \mathrm{e}^{t}(1-t).$$

$\square$

Although this result is less general than Lewandowski's and Szynal's [10], it was much easier to obtain. The main utility of this theorem is that it suggests one should focus on proving initial n^{th}-gap functions are extremal in a space where one suspects that it is true. Upon restricting a space to initial n^{th}-gap functions, one should be able to obtain some information as well.

The following lemma was discovered along the path of attempting to determine sufficient conditions for initial n^{th}-gap functions to be extremal in a space.

Lemma 3.0.2. *Let* $f(z) = (1+\alpha z)^\beta, \alpha \in \overline{B}(0,1)$, *and* $f^\alpha(z) = \sum_{k=0}^\infty b_k(\alpha) z^k$. *For every integer* k, $|b_k(\alpha)| \leq |\alpha b_k(1)|$ *if and only if* $\beta \in (-\infty, 0]$.

Proof. We shall use known results from Rainville's text [13] about Gegenbauer polynomials to determine $b_k(\alpha)$. Let $g(x,t,\alpha) = (1-2x+t^2)^{-\alpha}$. We can write $g(x,t,\alpha) = \sum_{k=0}^\infty C_k^\alpha(x) z^n$ whereas each $C_k^\alpha(x)$ is a Gegenbauer polynomial satisfying

$$C_{2k}^\alpha(0) = \frac{(-1)^k(\alpha)_k}{k!} \text{ and } C_{2k+1}^\alpha(0) = 0,$$

and $(x)_n = \prod_{j=1}^n (x+j-1)$.
Set $\omega = az$. Then,

$$f^\alpha(\omega^2) = (1+\omega^2)^{\alpha\beta} = g(0,\omega,-\alpha\beta) = \sum_{k=0}^\infty C_k^{-\alpha\beta}(0)\omega^k = \sum_{k=0}^\infty \frac{(-1)^{2k}(-\alpha\beta)_{2k}}{(2k)!}\omega^{2k}.$$

Hence,

$$f^\alpha(z) = \sum_{k=0}^\infty \frac{(-1)^k(-\alpha\beta)_k a^k z^k}{k!} \text{ and } b_k(\alpha) = \frac{(-1)^k(-\alpha\beta)_k a^k}{k!}.$$

Consequently, we have $|b_k(\alpha)| \leq |\alpha b_k(1)|$ if and only if $\left|\frac{(-1)^k(-\alpha\beta)_k a^k}{k!}\right| \leq \left|\frac{(-1)^k \alpha(-\beta) a^k}{k!}\right|$ if and only if $|(-\alpha\beta)_k| \leq |\alpha(-\beta)_k|$.

We now proceed by induction. For $k=1, |(-\alpha\beta)_k| = |-\alpha\beta| = |\alpha(\beta)_k|$. Assume $|(-\alpha\beta)_k| \leq |\alpha(-\beta)_k|$. Then, $|(-\alpha\beta)_{k+1}| = |(-\alpha\beta)_k(-\alpha\beta+k)| \leq |\alpha(-\beta)_k(-\alpha\beta+k)| \leq |\alpha(-\beta)_k(|\alpha\beta|+k)| \leq |\alpha(-\beta)_k(-\beta+k)| = |\alpha(-\beta)_{k+1}|$. For $\beta \notin (-\infty, 0)$, set $\alpha = \mathrm{e}^{-i\arg\beta}$ to see that $|(-\alpha\beta+k)| = |\beta|+k > |-\beta+k|$ for all $k \in \mathbb{N}$. □

Of course there appear to be many other functions satisfying this condition; however, the appropriate general setting has not yet been realized.

Let H be a space of nonvanishing, analytic functions whose domain is $\mathbb{D}$. Suppose there is an $f(z) = \sum_{k=0}^\infty a_k z^k$ such that $a_k \geq 0$ and $f \in M(H,n,0)$ for every $n \in \mathbb{N}$. Furthermore, suppose H^n contains an extremal function that is unique up to rotation. It would be useful if there were reasonable conditions that would guarantee H^{-1} has initial n^{th}-gap functions in $M(H^{-1},n,0)$. Such a theorem would tie $\mathcal{P}$, $\left[\mathcal{S}_0^*\right]^{-1}$, and $\mathcal{B}_0$ together since

$$\frac{1+z}{1-z} \in \mathcal{P}^{-1}, \frac{1}{(1-z)^2} \in S_0^*, \text{ and } \exp\left[\frac{1+z}{1-z}\right] \in \mathcal{B}_0^{-1}.$$

CHAPTER 4

BRANNAN'S CONJECTURE

Conjecture 4.0.3. *Let*

$$\frac{(1+xz)^{\alpha}}{(1-z)^{\beta}} = \sum_{k=0}^{\infty} A_k(\alpha,\beta,x)z^k \text{ for } \alpha,\beta \geq 0 \text{ and } |x| = 1.$$

Then $|A_n(\alpha,\beta,x)| \leq |A_n(\alpha,\beta,1)|$ *for all odd* $n \in \mathbb{N}$.

Let

$$\binom{x}{n} = \frac{1}{n!}\prod_{k=0}^{n-1}(x-k) \text{ for } n \geq 1 \text{ and } 0 \text{ for } n = 0.$$

Then,

$$\frac{(1+xz)^{\alpha}}{(1-z)^{\beta}} = \left(\sum_{k=0}^{\infty}\binom{\alpha}{k}(xz)^k\right)\left(\sum_{j=0}^{\infty}\binom{-\beta}{j}(-z)^j\right) =$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{\alpha}{j}(xz)^j\binom{-\beta}{k-j}(-z)^{k-j} = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k}(-1)^{k-j}\binom{\alpha}{j}x^j\binom{-\beta}{k-j}\right]z^k.$$

Hence,

$$A_n(\alpha,\beta,x) = \sum_{j=0}^{n}(-1)^{n-j}\binom{\alpha}{j}x^j\binom{-\beta}{n-j}.$$

Brannan's Conjecture has been proven for $\alpha = \beta \geq 1$ by Brannan, et al. [6] and for $\alpha \geq 1$ and $\beta = 1$ by Abaronov, et al. [2], which implies the case $\alpha, \beta \geq 1$. However, Brannan [5] showed that the conjecture could be true for at most odd n for α or β less than one. The case $\beta = 1$ has been established by Brannan [5] for ($n = 3$), Milctech [12] for ($n = 5$), and by Barnard, et al. [4] for ($n = 7$). The case $\alpha = \beta$ has recently been solved by Ruscheweyh and Salinas [15]. We shall be concerned with case $\beta = 1$ and discuss results of attempted methods, and give some observations that might lead to a proof.

4.1. Equivalent Statement

Let $\alpha > 0$ and $\theta \in [0, 2\pi]$. Since

$$A_n(\alpha, 1, x) = \sum_{j=0}^{n} (-1)^{n-j} \binom{\alpha}{j} x^j \binom{-1}{n-j} = \sum_{j=0}^{n} \binom{\alpha}{j} x^j,$$

$|A_n(\alpha, 1, e^{i\theta})| \leq |A_n(\alpha, 1, 1)|$ if and only if

$$\begin{aligned} 0 \leq |A_n(\alpha,1,1)|^2 - |A_n(\alpha,1,\mathrm{e}^{i\theta})|^2 &= \left[\sum_{k=0}^{n} \binom{\alpha}{k}^2 + 2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \binom{\alpha}{i}\binom{\alpha}{j} \right] \\ &- \left[\sum_{k=0}^{n} \binom{\alpha}{k}^2 \left[\cos^2 k\theta + \sin^2 k\theta\right] + 2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \binom{\alpha}{i}\binom{\alpha}{j} \left[\cos i\theta \cos j\theta + \sin i\theta \sin j\theta\right] \right] \\ &= 2 \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} \binom{\alpha}{i}\binom{\alpha}{j} (1 - \cos(j-i)\theta) = 2 \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} \binom{\alpha}{i}\binom{\alpha}{i+j} (1 - \cos j\theta) \\ &= 4 \sum_{j=1}^{n} \left[\sum_{i=0}^{n-j} \binom{\alpha}{i}\binom{\alpha}{i+j} \right] \sin^2(j\theta/2). \end{aligned}$$

Therefore, Brannan's Conjecture for $\beta = 1$ is equivalent to verifying

$$F_n(\alpha, \theta) := \sum_{j=1}^{n} \left[\sum_{i=0}^{n-j} \binom{\alpha}{i}\binom{\alpha}{i+j} \right] \sin^2(j\theta/2) \geq 0$$

for odd n and $\theta \in [0, \pi]$ due to the periodicity of $\sin^2(\theta/2)$.

During the rest of the paper, we will suppose that $\alpha \in (0, 1)$. The following identities will be used freely,

$$\binom{x}{n} = \frac{(-1)^n}{n!}(-x)_n, \quad (x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \quad (-x)_n = (-1)^n (x-n+1)_n,$$

$$(x+1)_n = \frac{(x+n)(x)_n}{x}, \quad (x-1)_n = \frac{(x-1)(x)_n}{x+n-1}, \quad (x)_{n+1} = (x+n)(x)_n,$$

$$(x)_{n-1} = \frac{(x)_n}{x+n-1}, \quad (x)_{k+j} = (x)_k (x+k)_j.$$

Let

$$f_{(j,k)}(\alpha) = \sum_{i=0}^{k-j} \binom{\alpha}{i}\binom{\alpha}{i+j}.$$

At first glance $f_{(j,k)}(\alpha)$ appears to have some nice properties such as monotonicity in α and having nice upper and lower bounds in ratios consisting of stepping up j

while fixing k. Even though neither of these hold, the sign of $f_{(j,k)}$ is rather easy to describe. Notice that for $\alpha \in (0,1)$ and $\theta \in (0,\pi)$, $\binom{\alpha}{i}\binom{\alpha}{i+j}$ is strictly positive or negative for any fixed j and k. Hence, we can construct $f^+_{(j,k)}(\alpha)$ from each of the positive terms of f and $f^-_{(j,k)}(\alpha)$ from all of the negative terms of f such that $f_{(j,k)}(\alpha) = f^+_{(j,k)}(\alpha) + f^-_{(j,k)}(\alpha)$. Since $\binom{x}{n}$ alternates in sign with respect to n, one can verify that

$$\operatorname{sign}\left(\binom{\alpha}{i}\binom{\alpha}{i+j}\right) = \begin{cases} 1 & , (i = 0 \text{ and } j \text{ is odd}) \text{ or } (i \neq 0 \text{ and } j \text{ is even}) \\ -1 & , (i = 0 \text{ and } j \text{ is even}) \text{ or } (i \neq 0 \text{ and } j \text{ is odd}) \end{cases}$$

which implies that

$$f^+_{(j,k)}(\alpha) = \begin{cases} \binom{\alpha}{j} & , j \text{ is odd} \\ \sum_{i=1}^{k-j}\binom{\alpha}{i}\binom{\alpha}{i+j} & , j \text{ is even} \end{cases} \quad f^-_{(j,k)}(\alpha) = \begin{cases} \sum_{i=1}^{k-j}\binom{\alpha}{i}\binom{\alpha}{i+j} & , j \text{ is odd} \\ \binom{\alpha}{j} & , j \text{ is even} \end{cases}$$

Lemma 4.1.1.

$$f_{(j,\infty)}(\alpha) := \sum_{i=0}^{\infty}\binom{\alpha}{i}\binom{\alpha}{i+j} = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+j+1)\Gamma(\alpha-j+1)}.$$

Proof.

$$\begin{aligned} \sum_{i=0}^{\infty}\binom{\alpha}{i}\binom{\alpha}{i+j} &= \sum_{i=0}^{\infty}\frac{(-1)^i(-\alpha)_i}{(1)_i}\frac{(-1)^{i+j}(-\alpha)_{i+j}}{(1)_{i+j}} = \\ &= (-1)^j\sum_{i=0}^{\infty}\frac{(-\alpha)_i}{(1)_i}\frac{(-\alpha)_j(-\alpha+j)_i}{(1)_j(1+j)_i} \\ &= \frac{(\alpha-j+1)_j}{(1)_j}\sum_{i=0}^{\infty}\frac{(-\alpha)_i(-\alpha+j)_i}{(1)_i(1+j)_i} \\ &= \frac{(\alpha-j+1)_j}{(1)_j}{}_2F_1(-\alpha,-\alpha+j;1+j,1) \end{aligned}$$

In general [1],

$$\begin{aligned} {}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b+1-c;1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1(c-a,c-b;1+c-a-b;1-z). \end{aligned}$$

Hence,

$$ {}_2F_1(a,b;c,1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. $$

and

$$ f_{(j,\infty)}(\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)} \left[\frac{\Gamma(1+j)\Gamma(2\alpha+1)}{\Gamma(\alpha+j+1)\Gamma(1+\alpha)} \right] $$

$$ \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+j+1)\Gamma(\alpha-j+1)} $$

□

It is straightforward to verify that

$$ \operatorname{sign}\left(\Gamma(x)\right) = \begin{cases} 1 & , x > 0 \text{ or } [x] \text{ is even} \\ -1 & , x < 0 \text{ and } [x] \text{ is odd} \end{cases} $$

which implies that

$$ \operatorname{sign}\left(\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+j+1)\Gamma(\alpha-j+1)} \right) = (-1)^{j+1}. $$

Notice that for j odd, $f_{(j,k)}(\alpha)$ is decreasing with respect to k and $f_{j,\infty}(\alpha)$ is positive. Hence, $f_{(j,k)}(\alpha)$ is positive for odd j and similarly negative for even j.

It seems likely that a proof of Brannan's Conjecture should entail a factoring of $\sin^2(\theta/2)$ based on the facts that $F_n(1,\theta) = \sin^2(\theta/2)$, $F_1(\alpha,\theta) = \alpha\sin^2(\theta/2)$, and the fact that each term of $F_n(\alpha,\theta)$ is divisible by $\sin^2(\theta/2)$. Such a factorization can be given by the following lemma:

Lemma 4.1.2.

$$ \frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)} = n^2{}_3F_2(1+n, 1-n, 1; 3/2, 2; \sin^2(\theta/2)). $$

Proof. We shall use the following facts from Askey's text [3]:

$$\frac{1-\cos(n+1)\theta}{2\sin\frac{1}{2}\theta} = \sum_{k=0}^{n}\sin(k+\frac{1}{2})\theta. \tag{4.1}$$

$$\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta} = \frac{P_n^{(1/2,-1/2)}(\cos\theta)}{P_n^{(-1/2,1/2)}(1)}. \tag{4.2}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}{}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;(1-x)/2). \tag{4.3}$$

$$P_n^{\alpha,\beta}(1) = \frac{(\alpha+1)_n}{n!}. \tag{4.4}$$

Therefore,

$$\begin{aligned}
\frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)} &= \frac{1}{\sin(\theta/2)}\frac{1-\cos n\theta}{2\sin(\theta/2)} \\
&\overset{4.1}{=} \sum_{k=0}^{n-1}\frac{\sin((k+\frac{1}{2})\theta)}{\sin(\theta/2)} \overset{4.2}{=} \sum_{k=0}^{n-1}\frac{P_k^{(1/2,-1/2)}(\cos\theta)}{P_k^{(-1/2,1/2)}(1)} \\
&\overset{4.4}{\underset{4.3}{=}} \sum_{k=0}^{n-1}\frac{(\frac{3}{2})_k/k!{}_2F_1(-k,k+1;\frac{3}{2},\frac{1-\cos\theta}{2})}{(\frac{1}{2})_k/k!} \\
&= \sum_{k=0}^{n-1}(2k+1)\sum_{m=0}^{k}\frac{(-k)_m(k+1)_m}{(\frac{3}{2})_m(1)_m}\sin^{2m}(\theta/2) \\
&= \sum_{m=0}^{n-1}\frac{\sin^{2m}(\theta/2)}{(\frac{3}{2})_m(1)_m}\sum_{k=m}^{n-1}(2k+1)(-k)_m(k+1)_m.
\end{aligned}$$

To proceed in the proof, we shall show the following:

$$\sum_{k=m}^{n-1}(2k+1)(-k)_m(k+1)_m = \frac{-(n)_{m+1}(-n)_{m+1}}{m+1}.$$

We proceed by induction. For $n=1$ we have $k=m=0$; hence,

$$(2k+1)(-k)_m(k+1)_m = 1 = \frac{-(n)_{m+1}(-n)_{m+1}}{m+1}.$$

Assuming true for n,

$$\sum_{k=m}^{n}(2k+1)(-k)_m(k+1)_m = \frac{-(n)_{m+1}(-n)_{m+1}}{m+1} + (2n+1)(-n)_m(n+1)_m$$

$$= \frac{-(n+1)_{m+1}(-n-1)_{m+1}}{m+1} \text{ if and only if}$$

$$(n)_{m+1}(-n)_{m+1} - (n+1)_{m+1}(-n-1)_{m+1} = (2n+1)(-n)_m(n+1)_m(m+1).$$

$$\begin{aligned}
&(n)_{m+1}(-n)_{m+1} - (n+1)_{m+1}(-n-1)_{m+1}\\
&= (n)_m[n+m](-n)_m[-n-m] - (n)_m\left[\frac{(n+m-1)(n+m)}{n}\right](-n)_m[-n-1]\\
&= (n)_m(-n)_m[(m^2-n^2) - \frac{1}{n}(n+m+1)(n+m)(-n-1)]\\
&= (-n)_m(n)_m(2n+1)[m+1]\frac{n+m}{n} = (2n+1)(-n)_m(n+1)_m(m+1).
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{\sin^2(n\theta/2)}{\sin^2(\theta/2)} &= \sum_{m=0}^{n-1}\frac{\sin^{2m}(\theta/2)}{(\frac{3}{2})_m(1)_m}\left[\frac{-(n)_{m+1}(-n)_{m+1}}{m+1}\right]\\
&= -\sum_{m=0}^{n-1}\frac{(n)_{m+1}(-n)_{m+1}}{(\frac{3}{2})_m(1)_{m+1}}\sin^{2m}(\theta/2).
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{n^2(n+1)_m(1-n)_m}{(\frac{3}{2})_m(2)_m} &= \frac{n^2}{(\frac{3}{2})_m(1)_{m+1}}\frac{(n+m+1)(n)_m}{n}\frac{-(-n+m+1)(-n)_m}{-n}\\
&= -\frac{(n)_{m+1}(-n)_{m+1}}{(\frac{3}{2})_m(1)_{m+1}},
\end{aligned}$$

and the lemma is shown. □

The following conjecture has the benefits of simplicity and numerical evidence; however, it is not immediately obvious how to generalize a proof for it.

Conjecture 4.1.3. *$\alpha^2\sin^2(\theta/2)/2 \le F_n(\alpha,\theta)$ for odd n and $\theta \in [0,\pi]$.*

Another approach the author has considered was to find a sufficient condition on the coefficient space of $a_1,\ldots,a_n$ for $f(\theta) = \sum_{j=1}^{k} a_j(1-\cos j\theta) \ge 0$ given that k is odd, $a_j \in \mathbb{R}$ for $j = 1,\ldots,k$, and $\theta \in [0,\pi]$. Such a condition might give direction in which one should attempt to prove Brannan's Conjecture.

Lemma 4.1.4. *If $k = 3$ and $a_1 = 1$, then $f(\theta) \geq 0$ for all $\theta \in [0, \pi]$ if and only if*

1. $a_2 \in [-1, \frac{2}{5}]$ *and* $a_3 \in [\frac{a_2^2}{4(1+a_2)}, \infty)$.
2. $a_2 \in [\frac{-2}{5}, 2]$ *and* $a_3 \in [\frac{-1}{9}(4a_2 + 1), \infty)$.
3. $a_2 \in [2, \infty)$ *and* $a_3 \in [-1, \infty)$.

Proof. It is clear that $(1 - \cos\theta) + a_2(1 - \cos 2\theta) + \frac{a^2}{4(1+a)}(1 - \cos 3\theta) = \frac{(1-\cos\theta)}{4(1+a)}(2 + 3a_2 + 2a_2 \cos\theta)^2 \geq 0$ for $a_2 \in [-1, \infty]$. For sharpness with $a_2 \in [-1, \frac{-2}{5}]$, it is sufficient to show that a_2 cannot be decreased. Hence, for our result to not be sharp there must be an $\epsilon > 0$ such that $(1-\cos\theta)+[a_2-\epsilon](1-\cos 2\theta)+\frac{a^2}{4(1+a)}(1-\cos 3\theta) \geq 0$. However, it can be shown that $(1 - \cos\theta) + [a_2 - \epsilon](1 - \cos 2\theta) + \frac{a^2}{4(1+a)}(1 - \cos 3\theta)$ evaluated at $\cos^{-1}(\frac{3a_2+2}{2a_2})$ is equal to $((a_2 + 2)(2 + 5a_2)\epsilon)/(2a_2^2)$ which is negative in $[-1, \frac{-2}{5}]$.

Writing $(1 - \cos\theta) + a_2(1 - \cos 2\theta) + a_3(1 - \cos 3\theta) = (\frac{1}{2} + 2a_2 + \frac{9a_3}{2})\theta^2 + O(\theta^3) = (2 + 2a_3) + O(\theta - \pi) = (\frac{3}{2} + \frac{3}{2}a_2) + O(\theta - \frac{2\pi}{3})$ as $\theta \to 0$ implies that is necessary for $\frac{1}{2} + 2a_2 + \frac{9a_3}{2} \geq 0$, $2 + 2a_3 \geq 0$, and $\frac{3}{2} + \frac{3}{2}a_2 \geq 0$. i.e $a_3 \in [\max(-1, \frac{-1}{9}(4a_2 + 1), \infty)$ and $a2 \geq -1$.

Finally, $(1 - \cos\theta) + a_2(1 - \cos 2\theta) - \frac{1}{9}(4a_2 + 1)(1 - \cos 3\theta) = \frac{2}{9}(1 - \cos\theta)^2((8a_2 + 2)\cos\theta + 4 + 7a_2) \geq 0$ for $a_2 \in [-\frac{2}{5}, 2]$ because in that range $|8a_2 + 2| \leq |4 + 7a_2|$. Once $a_3 = -1$ and $a_2 = 2$, we can fix a_3 and increase a_2 without decreasing the function since $1 - \cos 2\theta \geq 0$. □

There might be a generalizable sufficient condition; however, it has yet to be realized.

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